MATH 207(c)–Analysis IV

1. Solve the initial value problem

$$\begin{cases} \dfrac{d^4u}{dt^4} + \dfrac{d^2u}{dt^2} = e^{-t}, \\ u(0) = \dfrac{du}{dt}(0) = \dfrac{d^2u}{dt^2}(0) = \dfrac{d^3u}{dt^3}(0) = 0. \end{cases}$$

2. Consider the complex function

$$F(z) = \frac{z+1}{z^4 + 5z^2 + 4}.$$

(a) Compute the inverse Laplace transform of F using the integral formula (you can also verify that you got the correct result by alternatively computing the inverse Laplace transform using the properties of $\mathcal{L}[\cdot]$ and Laplace transforms of known examples).

(b) Restricting F on $\mathbb{R}$ (i.e. think of F as a function from $\mathbb{R}$ to $\mathbb{C}$), compute the inverse Fourier transform of F.

3. For the functions $f : [0,1] \to \mathbb{R}$ given below, compute its expansion into a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n\mathbb{N}\setminus\{0\}} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right).$$

(a) $f(x) = \sin^2(2\pi x)$,

(b) $f(x) = x\sin(2\pi x)$,

(c) $f(x) = e^{-x}$.

4. Consider the initial value problem for the modified heat equation on $x \in [0,1]$ for some $a \in \mathbb{R}$ with Dirichlet conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) = e^{-2x} \quad \text{for } t > 0,\ x \in (0,1), \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0 \quad \text{for } t > 0. \end{cases}$$

Find an expression for the solution u.

5. Let us consider the same initial value problem as in Exercise 4 but with Neumann boundary conditions instead (in the case of the heat equation, these model an *insulated* endpoint):

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) = e^{-2x} \quad \text{for } t > 0,\ x \in (0,1), \\ u(x,0) = 0, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 \quad \text{for } t > 0. \end{cases}$$

Find an expression for the solution u. *(Hint: You might want to extend your functions as* even *2-periodic functions in x before decomposing in Fourier modes.)*

6 (Extra). The n-moment of a function $f : [0,+\infty) \to \mathbb{C}$ is defined by

$$\mu_n = \int_0^{+\infty} t^n f(t)\, dt,$$

provided, of course, this integral converges. Show that, if all n-moments of f converge and

$$\sup_{n\in\mathbb{N}} \int_0^{+\infty} t^n |f(t)|\, dt < +\infty,$$

then

$$\mathcal{L}[f](z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \mu_n z^n.$$

In particular, it follows form the above that $\mathcal{L}[f](z)$ in this case extends holomorphically at $z = 0$ and the moments of f can be calculated in terms of the coefficients of the Taylor expansion of $\mathcal{L}[f](z)$ at $z = 0$.

Solutions

Problem 1

Laplace Transform

We apply the Laplace transform to the differential equation. We denote the Laplace transform of a function $u(t)$ by $\mathcal{L}[u(t)](z) = U(z)$. The Laplace transforms of the derivatives are:

$$\mathcal{L}\left[\frac{d^4u}{dt^4}\right](s) = s^4U(s) - s^3u(0) - s^2\frac{du}{dt}(0) - s\frac{d^2u}{dt^2}(0) - \frac{d^3u}{dt^3}(0) = s^4U(s),$$

$$\mathcal{L}\left[\frac{d^2u}{dt^2}\right] = s^2U(s) - su(0) - \frac{du}{dt}(0) = s^2U(s).$$

The Laplace transform of e^{-t} is:

$$\mathcal{L}[e^{-t}](s) = \frac{1}{s+1}.$$

Substituting these into the differential equation, we get:

$$s^4U(s) + s^2U(s) = \frac{1}{s+1}.$$

Factoring out $U(s)$, we have:

$$U(s)(s^4 + s^2) = \frac{1}{s+1}.$$

Solving for $U(s)$, we get:

$$U(s) = \frac{1}{(s+1)(s^4+s^2)} = \frac{1}{s^2(s^2+1)(s+1)}.$$

We decompose $U(s)$ into partial fractions:

$$\frac{1}{s^2(s^2+1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} + \frac{E}{s+1}.$$

Multiplying both sides by the denominator $s^2(s^2+1)(s+1)$, we get:

$$1 = As(s^2+1)(s+1) + B(s^2+1)(s+1) + (Cs+D)s^2(s+1) + Es^2(s^2+1).$$

Expanding and combining like terms, we have:

$$1 = As^4 + As^3 + As^2 + As + Bs^3 + Bs^2 + Bs + B + Cs^4 + Cs^3 + Ds^3 + Ds^2 + Es^4 + Es^2.$$

Grouping the terms by powers of s, we get:

$$1 = (A+C+E)s^4 + (A+B+C+D)s^3 + (A+B+D+E)s^2 + (A+B)s + B.$$

Equating the coefficients of corresponding powers of s on both sides, we obtain the system of equations:

$$\begin{aligned} A + C + E &= 0, \\ A + B + C + D &= 0, \\ A + B + D + E &= 0, \\ A + B &= 0, \\ B &= 1. \end{aligned}$$

Solving this system, we find:

$$B = 1,$$

$$A + 1 = 0 \implies A = -1,$$
$$-1 + C + E = 0 \implies C + E = 1,$$
$$-1 + 1 + C + D = 0 \implies C + D = 0,$$
$$-1 + 1 + D + E = 0 \implies D + E = 0.$$

From $C + D = 0$ and $D + E = 0$, we get $C = -D$ and $E = -D$. Substituting $C = -D$ into $C + E = 1$, we get:

$$-D - D = 1 \implies -2D = 1 \implies D = -\frac{1}{2}, C = -D = \frac{1}{2}, E = -D = \frac{1}{2}.$$

Thus, the partial fraction decomposition is:

$$U(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{s+1} = -\frac{1}{s} + \frac{1}{s^2} + \frac{s-1}{2(s^2+1)} + \frac{1}{2(s+1)}.$$

We now find the inverse Laplace transform of each term:

$$\mathcal{L}^{-1}\left[-\frac{1}{s}\right] = -1,$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t,$$
$$\mathcal{L}^{-1}\left[\frac{s-1}{2(s^2+1)}\right] = \frac{1}{2}\left(\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\right) = \frac{1}{2}(\cos t - \sin t),$$
$$\mathcal{L}^{-1}\left[\frac{1}{2(s+1)}\right] = \frac{1}{2}e^{-t}.$$

Combining these, we get the solution:

$$u(t) = -1 + t + \frac{1}{2}(\cos t - \sin t) + \frac{1}{2}e^{-t}.$$

Final Answer

$$\boxed{u(t) = -1 + t + \frac{1}{2}(\cos t - \sin t) + \frac{1}{2}e^{-t}}$$

Problem 2

Part (a)

Given the function:

$$F(z) = \frac{z+1}{z^4 + 5z^2 + 4},$$

the poles of $F(z)$ are the roots of the denominator $z^4+5z^2+4=0$. Factoring, we get:

$$z^4+5z^2+4=(z^2+1)(z^2+4)$$

So, the poles are $z=\pm i$ and $z=\pm 2i$. All of them are simple poles.

The inverse Laplace transform is given by the following integral (also known as Bromwich integral):

$$f(t)=\mathcal{L}^{-1}\{F(z)\}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}F(z)\,dz,$$

where $\gamma\in\mathbb{R}$ is chosen such that all the singularities of $F(z)$ lie to the left of the line $\mathrm{Re}(z)=\gamma$. In this case, we can choose any $\gamma>0$.

As we have seen this class, the way to compute this integral is schematically as follows: In order to compute $\int_{\gamma-iR}^{\gamma+iR} e^{zt}F(z)\,dz$ as $R\to+\infty$, we close the loop using a half circle to the left (i.e. the curve $\theta\to Re^{i\theta}$ for $\theta\in[\frac{\pi}{2},\frac{3\pi}{2}]$), and then we compute the integral over the closed loop using the reisdue theorem (note that this loop will contain, when R is large, all the poles of $e^{tz}F(z)$). As $R\to+\infty$, the integral over the half circle goes to 0 (this is why we chose to close the loop on the left, since e^{tz} is uniformly bounded in the region $\mathrm{Re}(z)\leqslant\gamma$ when $t\geqslant 0$), leaving us with

$$f(t)=\mathcal{L}^{-1}\{F(z)\}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}F(z)\,dz=\sum_{z_k:\text{ poles of }F}\mathrm{Res}_{z=z_k}\left(e^{tz}F(z)\right),$$

Residue Calculation

We calculate the residues at each pole.

- Residue at $z=i$:

$$\mathrm{Res}_{z=i}(e^{zt}F(z))=\lim_{z\to i}(z-i)e^{zt}\frac{z+1}{(z^2+1)(z^2+4)}=e^{it}\frac{i+1}{(2i)(3)}=\frac{e^{it}(i+1)}{6i}$$

- Residue at $z=-i$:

$$\mathrm{Res}_{z=-i}(e^{zt}F(z))=\lim_{z\to -i}(z+i)e^{zt}\frac{z+1}{(z^2+1)(z^2+4)}=e^{-it}\frac{-i+1}{(-2i)(3)}=\frac{e^{-it}(i-1)}{6i}$$

- Residue at $z=2i$:

$$\mathrm{Res}_{z=2i}(e^{zt}F(z))=\lim_{z\to 2i}(z-2i)e^{zt}\frac{z+1}{(z^2+1)(z^2+4)}=e^{2it}\frac{2i+1}{(-3)(4i)}=\frac{e^{2it}(2i+1)}{-12i}$$

- Residue at $z=-2i$:

$$\mathrm{Res}_{z=-2i}(e^{zt}F(z))=\lim_{z\to -2i}(z+2i)e^{zt}\frac{z+1}{(z^2+1)(z^2+4)}=e^{-2it}\frac{-2i+1}{(-3)(-4i)}=\frac{e^{-2it}(-2i+1)}{12i}$$

Summing the residues, we get:

$$f(t) = \left(\frac{e^{it}(i+1)}{6i} + \frac{e^{-it}(i-1)}{6i} + \frac{e^{2it}(2i+1)}{-12i} + \frac{e^{-2it}(-2i+1)}{12i} \right)$$

Combining the terms and using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we finally obtain:

$$f(t) = \frac{1}{3}\cos t + \frac{1}{3}\sin t - \frac{1}{3}\cos 2t - \frac{1}{6}\sin 2t$$

Part (b)

We compute the inverse Fourier transform of

$$F(\xi) = \frac{\xi+1}{(\xi^2+1)(\xi^2+4)},$$

so

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi x} \frac{\xi+1}{(\xi^2+1)(\xi^2+4)}\, d\xi.$$

The integrand

$$G_x(\xi) = e^{i\xi x} \frac{\xi+1}{(\xi^2+1)(\xi^2+4)}$$

has simple poles at $\xi = \pm i$ and $\xi = \pm 2i$.

We have seen how to compute the above integral before, as an application of the residue theorem. We need to distinugish two cases, based on the sign of x:

- When $x \geqslant 0$, the function $e^{ix\xi}$ is bounded on the upper half plane $\mathrm{Im}(\xi) \geqslant 0$. In this case, in order to compute the integral $\int_{-R}^{+R} e^{i\xi x} \frac{\xi+1}{(\xi^2+1)(\xi^2+4)}\, d\xi$ as $R \to +\infty$, we close the loop with a half circle on the *upper* half plane, namely the half circle $\theta \to Re^{i\theta}$, $\theta \in [0,\pi]$. This loop is counter-clockwise oriented and will include only the poles at $+i$ and $+2i$. So applying the residue theorem (and using the fact that the integral over the half circle goes to 0 as $R \to +\infty$), we get:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G_x(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} 2\pi i \left(\mathrm{Res}_{z=+i}(G_x(z)) + \mathrm{Res}_{z=+2i}(G_x(z)) \right).$$

  Computing the residues similarly as in the case of part (a), we obtain for $x \geqslant 0$:

$$f(x) = \sqrt{2\pi} i \left(\frac{e^{-x}(i+1)}{6i} + \frac{e^{-2x}(2i+1)}{-12i} \right) = 2\pi \left(\frac{1+i}{6} e^{-x} - \frac{1+2i}{12} e^{-2x} \right).$$

- When $x < 0$, the function $e^{ix\xi}$ is bounded on the lower half plane $\mathrm{Im}(\xi) \leqslant 0$. In this case, in order to compute the integral $\int_{-R}^{+R} e^{i\xi x} \frac{\xi+1}{(\xi^2+1)(\xi^2+4)}\, d\xi$ as $R \to +\infty$, we close the loop with a half circle on the *lower* half plane, namely following the half circle $\theta \to Re^{-i\theta}$, $\theta \in [0,\pi]$. This loop is *clockwise oriented* and will include only the poles at $-i$ and $-2i$. So applying the residue theorem (and using the fact that the integral over the half circle goes to 0 as $R \to +\infty$), we get (note the minus sign coming from the clockwise orientation):

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G_x(\xi)\, d\xi = -\frac{1}{\sqrt{2\pi}} 2\pi i \left(\mathrm{Res}_{z=-i}(G_x(z)) + \mathrm{Res}_{z=-2i}(G_x(z))\right).$$

Computing the residues similarly as before, we obtain for $x < 0$ (note in this case $x = -|x|$):

$$f(x) = -\sqrt{2\pi} i \left(\frac{e^x(i-1)}{6i} + \frac{e^{2x}(-2i+1)}{12i} \right) = 2\pi \left(\frac{1-i}{6} e^{-|x|} + \frac{-1+2i}{12} e^{-2|x|} \right).$$

So, overall:

$$\mathcal{F}^{-1}[F](x) = \begin{cases} 2\pi \left(\frac{1+i}{6} e^{-x} - \frac{1+2i}{12} e^{-2x} \right), & x \geqslant 0, \\ 2\pi \left(\frac{1-i}{6} e^{-|x|} - \frac{1-2i}{12} e^{-2|x|} \right), & x \leqslant 0. \end{cases}$$

Problem 3

For each of the following functions $f : [0,1] \to \mathbb{R}$, compute its Fourier series expansion in the form:

$$f(x) = \frac{1}{2} a_0 + \sum_{n \in \mathbb{N} \setminus \{0\}} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right)$$

(a) $f(x) = \sin^2(2\pi x)$

Use the trigonometric identity:

$$\sin^2(2\pi x) = \frac{1 - \cos(4\pi x)}{2}$$

So, the Fourier series is:

$$f(x) = \frac{1}{2} - \frac{1}{2} \cos(4\pi x)$$

Hence:

$$a_0 = 1, \quad a_2 = -1, \quad a_n = 0 \text{ for } n \neq 2, \quad b_n = 0 \text{ for all } n$$

(b) $f(x) = x \sin(2\pi x)$

We compute the Fourier coefficients:

Cosine coefficients a_n: Using the identity:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

we compute:

$$a_n = 2\int_0^1 x\sin(2\pi x)\cos(2\pi nx)\,dx = \int_0^1 x\left[\sin(2\pi(n+1)x) + \sin(2\pi(1-n)x)\right]dx$$

Using:

$$\int_0^1 x\sin(2\pi kx)\,dx = -\frac{1}{2\pi k}$$

(which follows simply by integrating by parts in $\sin(2\pi kx) = -\frac{1}{2\pi k}\frac{d}{dx}\cos(2\pi kx)$), we get:

$$a_n = -\frac{1}{\pi}\left(\frac{1}{n+1} + \frac{1}{1-n}\right) = \frac{2n}{\pi(n^2-1)} \quad \text{for } n \neq 1$$

For $n = 1$:

$$a_1 = \int_0^1 x\sin(4\pi x)\,dx = -\frac{1}{4\pi}$$

Sine coefficients b_n: Using:

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

we find:

$$b_n = 2\int_0^1 x\sin(2\pi x)\sin(2\pi nx)\,dx = \int_0^1 x\left[\cos(2\pi(n-1)x) - \cos(2\pi(n+1)x)\right]dx$$

Since:

$$\int_0^1 x\cos(2\pi kx)\,dx = 0 \text{ for integer } k \neq 0$$

we have $b_n = 0$ for $n \neq 1$. For $n = 1$:

$$b_1 = 2\int_0^1 x\sin^2(2\pi x)\,dx = \int_0^1 x(1-\cos(4\pi x))\,dx = \frac{1}{2}$$

Final Fourier series for (b):

$$f(x) = x\sin(2\pi x) = \frac{a_0}{2} + \sum_{n\geqslant 1} a_n\cos(2\pi nx) + \frac{1}{2}\sin(2\pi x)$$

with:

$$a_1 = -\frac{1}{4\pi}, \quad a_n = \frac{2n}{\pi(n^2-1)} \text{ for } n \neq 1, \quad b_n = 0 \text{ for } n \neq 1$$

(c) $f(x) = e^{-x}$

Zeroth coefficient:

$$a_0 = \int_0^1 e^{-x}\,dx = 1 - e^{-1}$$

Cosine coefficients:

$$a_n = 2\int_0^1 e^{-x}\cos(2\pi n x)\,dx = \frac{2(1-e^{-1})}{1+(2\pi n)^2}$$

Sine coefficients:

$$b_n = 2\int_0^1 e^{-x}\sin(2\pi n x)\,dx = \frac{4\pi n(1-e^{-1})}{1+(2\pi n)^2}$$

Final Fourier series for (c):

$$f(x) = \frac{1-e^{-1}}{2} + \sum_{n=1}^{\infty}\left[\frac{2(1-e^{-1})}{1+(2\pi n)^2}\cos(2\pi n x) + \frac{4\pi n(1-e^{-1})}{1+(2\pi n)^2}\sin(2\pi n x)\right]$$

Problem 4

Consider the initial value problem for the modified heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) = e^{-2x}, & t>0,\ x\in(0,1) \\ u(x,0) = 0, & x\in(0,1) \\ u(0,t) = u(1,t) = 0, & t>0 \end{cases}$$

Separation of Variables and Expansion

We seek a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$$

(this form satisfies the homogeneous Dirichlet boundary conditions; the justification for seeking such an expression, as usual, is that we extend our functions as odd, 2-periodic functions of $x \in \mathbb{R}$, such functions have only the sinuses terms present in the corresponding trigonometric expansion).

After extending the source term $f(x) = e^{-2x}$ as an odd, 2-periodic function of x, we expand it in a sine series:

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x), \quad f_n = 2\int_0^1 e^{-2x}\sin(n\pi x)\,dx$$

The integral can be computedeasily, by integrating by parts twice:

$$f_n = 2\int_0^1 e^{-2x}\sin(n\pi x)\,dx = \frac{2\pi n}{\pi^2 n^2 + 4}\left(1 - e^{-2}\cos(\pi n)\right)$$

Substitute into the PDE

We substitute into the PDE:

$$\sum_{n=1}^{\infty} b_n'(t)\sin(\pi n x) + \sum_{n=1}^{\infty} b_n(t)\pi^2 n^2 \sin(\pi n x) - a\sum_{n=1}^{\infty} b_n(t)\sin(\pi n x) = \sum_{n=1}^{\infty} f_n \sin(\pi n x)$$

This gives for each n:

$$b_n'(t) + (\pi^2 n^2 - a)b_n(t) = f_n.$$

Recall also that, since our initial condition was $u(x,0) = 0$, we have

$$b_n(0) = 0.$$

Solving the above (and noting that f_n is constant in t, since f was constant in t), we get

$$b_n(t) = \int_0^t e^{-(\pi^2 n^2 - a)(t-s)} f_n \, ds = f_n \cdot \frac{1 - e^{-(\pi^2 n^2 - a)t}}{\pi^2 n^2 - a}$$

Putting everything together:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2\pi n}{\pi^2 n^2 + 4}\left(1 - e^{-2}\cos(\pi n)\right) \cdot \frac{1 - e^{-(\pi^2 n^2 - a)t}}{\pi^2 n^2 - a} \cdot \sin(\pi n x)$$

Problem 5

Fourier Cosine Series Expansion

Since the boundary conditions are Neumann, we choose to extend our function $u(x,t)$ and our source term $f(x) = e^{-2x}$ as **even**, 2-periodic functions of x. This is because a C^1, even and 2 periodic function $h(x)$ will automatically satisfy the Neumann boundary conditions $h'(0) = 0 = h'(1)$. In this case, the trigonometric expansion of those functions will only contain cosine terms:

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

(note that the basis elements, namely $\cos(n\pi x)$, automatically satisfy the Neumann conditions at the boundary).

The source term $f(x) = e^{-2x}$, when extended as above, is expanded in a cosine series as follows:

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos\left(\frac{n\pi x}{L}\right)$$

with coefficients:

$$f_n = 2\int_0^1 e^{-2x}\cos(n\pi x)\,dx \quad \text{for} \quad n \geqslant 0.$$

Using the series for $u(x,t)$ and $f(x)$, we substitute into the PDE and equate coefficients:

$$\frac{1}{2}\left(a_0'(t) - aa_0(t)\right) + \sum_{n=1}^{\infty}\Big(a_n'(t) + (\pi^2 n^2 - a)a_n(t)\Big)\cos\left(\pi n x\right) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos\left(\pi n x\right).$$

Equating the corresponding coeffieicents of $\cos(n\pi x)$ for any $n \geqslant 0$, we get

$$a_n'(t) + (\pi^2 n^2 - a)a_n(t) = f_n.$$

Since our initial condition is that $u(x,0) = 0$, we have

$$a_n(0) = 0.$$

Solving the above first-order ODE, we get:

$$a_n(t) = \frac{f_n}{\pi^2 n^2 - a}\left(1 - e^{-(\pi^2 n^2 - a)t}\right).$$

Problem 6 (Extra)

The n-moment of a function $f : [0, +\infty) \to \mathbb{C}$ is defined by:

$$\mu_n = \int_0^{\infty} t^n f(t)\, dt,$$

provided this integral converges.

Assume that all n-moments of f converge and

$$\sup_{n\in\mathbb{N}} \int_0^{\infty} t^n |f(t)|\, dt < +\infty.$$

We aim to prove that:

$$\mathcal{L}[f](z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mu_n z^n.$$

The Laplace transform of f is defined as:

$$\mathcal{L}[f](z) = \int_0^{\infty} e^{-zt} f(t)\, dt.$$

For any $z \in \mathbb{C}$, the Taylor expansion of e^{-zt} reads:

$$e^{-zt} = \sum_{n=0}^{\infty} \frac{(-z)^n t^n}{n!}.$$

Substituting the above expression in the Laplace transform, we get

$$\mathcal{L}[f](z) = \int_0^{+\infty} \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!} t^n f(t)\, dt.$$

Interchanging the sum with the integral, we get

$$\mathcal{L}[f](z) = \sum_{n=0}^{+\infty} \int_0^{+\infty} \frac{(-z)^n}{n!} t^n f(t)\, dt = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!} \int_0^{+\infty} t^n f(t)\, dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \mu_n z^n,$$

which is the required result.

Remark: The reason that we were able to exchange the sum and the integral was precisely our assumption that

$$\sup_{n\in\mathbb{N}} \int_0^{\infty} t^n |f(t)|\, dt < +\infty.$$

This allows us to apply the so-called dominated convergence theorem. Moreover, in this case, since $\sup_n \mu_n < +\infty$, the series $\mathcal{L}[f](z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \mu_n z^n$ has infinite radius of analyticity; therefore, $\mathcal{L}[f](z)$ is defined on the whole of the complex plane.